ON A SUBCLASS OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract— In this paper, we introduce and study a new subclass of multivalent functions with negative coefficients in the unit disc. By giving specific values of we obtain important classes studied by various researchers in earlier work. The result presented by here included coefficient estimates. of several functions belonging to this class.

Keywords: Analytic, Salagean operator and Mmultivalent functions.

I. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z^p + \sum_{n=2}^{\infty} a_{k+p-1} z^{k+p-1}, \qquad (1.1)$$

which are analytic in the unit disc $U = \{Z \in C : |Z| < 1\}$

Let D^n be the Salagean operator [9] See also [10] $D^n : A \to A, n \in N,$ defined as-

$$D^0\ f(z) = f(z)$$

$$D^1\ f(z) = D\ f(z) = z\ f'(z)$$

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$$D^n\ f(z) = D(D^{n-1}\ f(z))$$

$$\left(n \in N_0 = N \cup \{0\}\right)$$

Given two functions $f(z), g(z) \in A,$ where $f(z)$ is given by (1.1) and $g(z)$ is defined by-

$$g(z) = z^p + \sum_{n=2}^{\infty} a_{k+p-1} z^{k+p-1} \qquad (1.2)$$

$$(f * g)(z) = z^p + \sum_{n=2}^{\infty} a_{k+p-1} b_{k+p-1} z^{k+p-1}$$

Again, let $\phi(z) = z^p + \sum_{n=2}^{\infty} \lambda_{k+p-1} z^{k+p-1}$

$$\psi(z) = z^p + \sum_{n=2}^{\infty} \mu_{k+p-1} z^{k+p-1}$$

And

Analytic in $U = \{Z \in C : |Z| < 1\}$ with $\lambda_{k+p-1} \geq 0, \mu_{k+p-1} \geq 0$ and $\lambda_{k+p-1} \geq \mu_{k+p-1}$.

We note that-

$$(f * \phi)(z) = z^p + \sum_{n=2}^{\infty} (k+p-1)^m a_{k+p-1} z^{k+p-1} \quad (1.3)$$

$$(f * \psi)(z) = z^p + \sum_{n=2}^{\infty} (k+p-1)^n \mu_{k+p-1} z^{k+p-1} \quad (1.4)$$

Using the binomial expansion on (1.3) and (1.4), we have the following-

$$\gamma(z) = z^p + \sum_{n=2}^{\infty} a_{k+p-1}(\beta) \lambda_{k+p-1}(\beta) z^{k+p+\beta-2} \quad (1.5)$$

$$\eta(z) = z^p + \sum_{n=2}^{\infty} a_{k+p-1}(\beta) \mu_{k+p-1}(\beta) z^{k+p+\beta-2} \quad (1.6)$$

Where $\lambda_{k+p-1}(\beta), \mu_{k+p-1}(\beta), a_{k+p-1}(\beta)$ are coefficients $\lambda_{k+p-1}, \mu_{k+p-1}, a_{k+p-1}$ are respectively depending β.

A function $f(z) \in A$ is said to be in the class $E_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$ if and only if

$$\frac{D^m \gamma(z)}{D^n \eta(z)} \prec (1-\alpha) \frac{1+Az}{1+Bz} + \alpha \ldots. \quad (1.7)$$

Where $\prec$ denote the subordination, where A and B are arbitrary fixed numbers, $-1 \leq B < A \leq 1, -1 \leq B < 0$. In other words $f(z) \in E_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$ iff there exists an analytic function $w(z)$ satisfies $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$ such that

$$\frac{D^m \gamma(z)}{D^n \eta(z)} \prec (1-\alpha) \frac{1+Aw(z)}{1+Bw(z)} + \alpha. \quad (1.8)$$

This is equivalent to-

$$\left| \frac{D^m \gamma(z)}{D^n \eta(z)} - 1 \right| < \left| (A-B)(1-\alpha) - B\left(\frac{D^m \gamma(z)}{D^n \eta(z)} - 1 \right) \right|, z \in U \quad (1.9)$$

Let τ denote the subclass of A whose elements can be expressed in the form-

$$f(z) = z^p - \sum_{n=2}^{\infty} a_{k+p-1} z^{k+p-1}, a_{k+p-1} \geq 0,$$

We shall denote by $\tilde{E}_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$, the subclass of functions in $E_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$ that have their non-zero Coefficient from second onwards, all negative, thus, we have-

$$\tilde{E}_{m,n}(\gamma, \eta; A, B, \alpha, \beta) = E_{m,n}(\gamma, \eta; A, B, \alpha, \beta) \cap T$$

By giving a specific values of $\gamma, \eta, A, B, \alpha$ and β , we obtain imported classes studied by various researchers in earlier works (see [2], [3], [5], [6], [7], [11], [12]).

In view of this remark we see that a study of the class $\widetilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$ leads to unified results on properties of various subclasses of multivalent function.

2. COEFFICIENT INEQUALITIES

In this section, we give some result for the class. Our first result is contained in the following theorem.

Theorem 2.1 If $f(z)\in A$ satisfies

$$\sum_{k=2}^{\infty}\Big\{(1-\beta)\left[(p\beta)^m-(p\beta)^m+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right]$$

$$+(A-B)(1-\alpha)(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\Big]\Big\}\left|a_{k+p-1}(\beta)\right|\le(A-B)(1-\alpha)\beta^{k+p-1}, \tag{2.1}$$

For some $\lambda_{k+p-1}(\beta)\ge 0,\ \mu_{k+p-1}(\beta)\ge 0,\ \lambda_{k+p-1}(\beta)\ge\mu_{k+p-1}(\beta),\ \alpha\,(0\le\alpha<1),\beta,m\in N$

and $n\in N_0$ then $f(z)\in E_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$

Proof Let condition (2.1) holds, then we have

$$\left|D^m\gamma(z)-D^n\eta(z)\right|-\left|(A-B)(1-\alpha)D^n\eta(z)-B\left[D^m\gamma(z)-D^n\eta(z)\right]\right|$$

$$=\Big|(p\beta)^m z^{p\beta}+\sum_{k=2}^{\infty}(k+p-2)^m a_{k+p-1}(\beta)\lambda_{k+p-1}(\beta)z^{k+p+\beta-2}-\Big[(p\beta)^n z^{p\beta}$$

$$+\sum_{k=2}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\,\mu_{k+p-1}(\beta)z^{k+p+\beta-2}\Big]-\Big|(A-B)(1-\alpha)\Big[(p\beta)^m z^{p\beta}$$

$$+\sum_{k=2}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\,\mu_{k+p-1}(\beta)z^{k+p+\beta-2}\Big]-B\Big\{\Big[(p\beta)^m z^{p\beta}$$

$$+\sum_{k=2}^{\infty}(k+p+\beta-2)^m a_{k+p-1}(\beta)\,\mu_{k+p-1}(\beta)z^{k+p+\beta-2}\Big]-\Big[(p\beta)^n z^{p\beta}$$

$$+\sum_{k=2}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\,\mu_{k+p-1}(\beta)z^{k+p+\beta-2}\Big]$$

$$=\Big|z^{p\beta}\left((p\beta)^m-(p\beta)^n\right)+\sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right]a_{k+p-1}(\beta)z^{k+p+\beta-2}\Big|$$

$$-\Big|(A-B)(1-\alpha)(p\beta)^n z^{p\beta}+(A-B)(1-\alpha)\times\sum_{k=2}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\,\mu_{k+p-1}(\beta)\,z^{k+p+\beta-2}\Big|$$

$$-B\left\{z^{p\beta}\left((p\beta)^m-(p\beta)^n\right)+\sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right]a_{k+p-1}(\beta)z^{k+p+\beta-2}\right\}$$

$$\le|z|^{p\beta}\left((p\beta)^m-(p\beta)^n\right)+\sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right]\left|a_{k+p-1}(\beta)\right||z|^{k+p+\beta-2}$$

$$-(A-B)(1-\alpha)(p\beta)^n|z|^{p\beta}+(A-B)(1-\alpha)\times\sum_{k=2}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\,\mu_{k+p-1}(\beta)|z|^{k+p+\beta-2}$$

$$-B\left\{|z|^{p\beta}\left((p\beta)^m-(p\beta)^n\right)+\sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right]a_{k+p-1}(\beta)\right\}|z|^{k+p+\beta-2}$$

$$\leq\left[(1-B)\left((p\beta)^m-(p\beta)^n\right)+\sum_{k=2}^{\infty}\{(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right.$$

$$\left.+(A-B)(1-\alpha)\times\sum_{k=2}^{\infty}(k+p+\beta-2)^n\ \ \mu_{k+p-1}(\beta)\right]\left|a_{k+p-1}(\beta)\right|-(A-B)(1-\alpha)(p\beta)^n$$

$$\leq\sum_{k=2}^{\infty}\left[(1-B)\left((p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\times\mu_{k+p-1}(\beta)\right)\right.$$

$$\left.+(A-B)(1-\alpha)\sum_{k=2}^{\infty}(k+p+\beta-2)^n\ \ \mu_{k+p-1}(\beta)\right]\times\left|a_{k+p-1}(\beta)\right|-(A-B)(1-\alpha)(p\beta)^n\leq 0$$

This completes the proof theorem (2.1)

Theorem 2.2 Let $f(z)\in\tau$ (as defined above) $f(z)\in\widetilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$ iff

$$\sum_{k=2}^{\infty}\left[(1-B)\left((p\beta)^m-(p\beta)^n\right)+(1-B)\left\{(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right\}\right.$$
$$\left.+(A-B)(1-\alpha)(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right]a_{k+p-1}(\beta)\leq(A-B)(1-\alpha)(p\beta)^n$$

For some $\lambda_{k+p-1}(\beta)\geq 0,\ \mu_{k+p-1}(\beta)\geq 0,\ \lambda_{k+p-1}(\beta)\geq\mu_{k+p-1}(\beta),\ \alpha\,(0\leq\alpha<1),\beta,m\in N$

and $n\in N_0$

Proof Since $\widetilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)\subset E_{m,n}(\gamma,\eta;A,B,\alpha,\beta),$ we only to prove the only part of the theorem for function $f(z)\in\tau,$ we can write

$$\left|\frac{\dfrac{D^m\gamma(z)}{D^n\eta(z)}-1}{(A-B)(1-\alpha)-B\left(\dfrac{D^m\gamma(z)}{D^n\eta(z)}-1\right)}\right|$$

$$=\left|\frac{z^{p\beta}\left((p\beta)^m-(p\beta)^n\right)+\sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right]a_{k+p-1}(\beta)\,z^{k+p+\beta-2}}{\begin{array}{l}(A-B)(1-\alpha)(p\beta)^n z^{p\beta}+(A-B)(1-\alpha)\times\sum_{k=2}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\ \mu_{k+p-1}(\beta)z^{k+p+\beta-2}\\ -B\left\{\left((p\beta)^m-(p\beta)^n\right)+\sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right]a_{k+p-1}(\beta)\,z^{k+p+\beta-2}\right\}\end{array}}\right|$$

$$= \left| \frac{\left[z^{p\beta}(p\beta)^m + \sum_{k=2}^{\infty}(k+p+\beta-2)^m \lambda_{k+p-1}(\beta)\, a_{k+p-1}(\beta) z^{k+p+\beta-2}\right] - \left[(p\beta)^n z^{p\beta} + \sum_{k=2}^{\infty}(k+p+\beta-2)^n \mu_{k+p-1}(\beta) a_{k+p-1}(\beta) z^{k+p+\beta-2}\right]}{(A-B)(1-\alpha)\left[(p\beta)^n z^{p\beta} + \sum_{k=0}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\, \mu_{k+p-1}(\beta) z^{k+p+\beta-2}\right] - \left[B(p\beta)^m z^{p\beta} + \sum_{k=2}^{\infty}(k+p+\beta-2)^m \lambda_{k+p-1}(\beta)\, a_{k+p-1}(\beta) z^{k+p+\beta-2}\right] + B\left[(p\beta)^n z^{p\beta} + \sum_{k=2}^{\infty}[(k+p+\beta-2)^n \mu_{k+p-1}(\beta) a_{k+p-1}(\beta) z^{k+p+\beta-2}\right]} \right|$$

$$= \left| \frac{z^{p\beta}\left((p\beta)^m - (p\beta)^n\right) + \sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m \lambda_{k+p-1}(\beta) - (k+p+\beta-2)^n \mu_{k+p-1}(\beta)\right] a_{k+p-1}(\beta) z^{k+p+\beta-2}}{(A-B)(1-\alpha)(p\beta)^n z^{p\beta} + (A-B)(1-\alpha)\sum_{k=2}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\, \mu_{k+p-1}(\beta) z^{k+p+\beta-2} - B\left((p\beta)^m - (p\beta)^n\right) z^{p\beta} + \sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m \lambda_{k+p-1}(\beta) - (k+p+\beta-2)^n \mu_{k+p-1}(\beta)\right] a_{k+p-1}(\beta) z^{k+p+\beta-2}} \right|$$

$$= \left| \frac{\left((p\beta)^m - (p\beta)^n\right) + \sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m \lambda_{k+p-1}(\beta) - (k+p+\beta-2)^n \mu_{k+p-1}(\beta)\right] a_{k+p-1}(\beta) z^{k+p+\beta-2-p\beta}}{(p\beta)^n (A-B)(1-\alpha) + (A-B)(1-\alpha)\sum_{k=2}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\, \mu_{k+p-1}(\beta) z^{k+p+\beta-p\beta-2} - B\left((p\beta)^m - (p\beta)^n\right) z^{p\beta} + \sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m \lambda_{k+p-1}(\beta) - (k+p+\beta-2)^n \mu_{k+p-1}(\beta)\right] a_{k+p-1}(\beta) z^{k+p+\beta-2-p\beta}} \right|$$

$\operatorname{Re}(z) \le |z|, \forall z \in U,$

$$= \left\{ \frac{\left((p\beta)^m - (p\beta)^n\right) + \sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m \lambda_{k+p-1}(\beta) - (k+p+\beta-2)^n \mu_{k+p-1}(\beta)\right] a_{k+p-1}(\beta) z^{k+p+\beta-2-p\beta}}{(p\beta)^n (A-B)(1-\alpha) - (A-B)(1-\alpha) \sum_{k=2}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\, \mu_{k+p-1}(\beta) z^{k+p+\beta-p\beta-2} + B\left[(p\beta)^m - (p\beta)^n + \sum_{k=2}^{\infty}\left((k+p+\beta-2)^m \lambda_{k+p-1}(\beta) - (k+p+\beta-2)^n \mu_{k+p-1}(\beta)\right) a_{k+p-1}(\beta) z^{k+p+\beta-2-p\beta}\right]} \right\} < 1$$

If we choose z real and letting $z \to 1^{-}$, we have

$$=\left((p\beta)^m-(p\beta)^n\right)+\sum_{k=2}^{\infty}\left[(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right]a_{k+p-1}(\beta)$$

$$\le (p\,\beta)^n\,(A-B)(1-\alpha)-(A-B)(1-\alpha)\sum_{k=2}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\ \ \mu_{k+p-1}(\beta)$$

$$+B\left[(p\beta)^m-(p\beta)^n+\sum_{k=2}^{\infty}\left((k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right)a_{k+p-1}(\beta)\right]$$

$$=\sum_{k=2}^{\infty}(1-B)\left[(p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right]a_{k+p-1}(\beta)$$

$$+(A-B)(1-\alpha)\sum_{k=2}^{\infty}(k+p+\beta-2)^n a_{k+p-1}(\beta)\ \ \mu_{k+p-1}(\beta)\le (A-B)(1-\alpha)\,(p\beta)^n$$

We can written as

$$=\sum_{k=2}^{\infty}(1-B)\left((p\beta)^m-(p\beta)^n\right)+(1-B)\left[(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right.$$

$$\left.+(A-B)(1-\alpha)\,(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right]a_{k+p-1}(\beta)\ \le (A-B)(1-\alpha)\ (p\beta)^n$$

To prove our next result, we shall need the following theorem.

Corollary 2.1 Let $f(z)\in\tau$ as defined above and $f(z)\in\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$ then,

$$a_{k+p-1}(\beta)\le\frac{(A-B)(1-\alpha)\ (p\beta)^n}{(1-B)\left[(p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right]+(A-B)(1-\alpha)\,(k+p+\beta-2)^n\mu_{k+p-1}(\beta)}$$

Proof Since $f(z)\in\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$

$$\left|a_{k+p-1}(\beta)\right|\le\frac{(A-B)(1-\alpha)\ (p\beta)^n}{(1-B)\left[(p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right]+(A-B)(1-\alpha)\,(k+p+\beta-2)^n\mu_{k+p-1}(\beta)}$$

This completes the proof.

3 Extreme points of the class

$\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$

Theorem 3.1 $f_p(z)\in z^p$

$$f_{k+p-1}(z)\in z^p-\frac{(A-B)(1-\alpha)\ (p\beta)^n}{(1-B)\left[(p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\ \mu_{k+p-1}(\beta)\right]+(A-B)(1-\alpha)\,(k+p+\beta-2)^n\mu_{k+p-1}(\beta)}\,z^{k+p+\beta-2}$$

For some $\lambda_{k+p-1}(\beta)\geq 0$, $\mu_{k+p-1}(\beta)\geq 0$, $\lambda_{k+p-1}(\beta)\geq \mu_{k+p-1}(\beta)$, $\alpha\,(0\leq\alpha<1)$, $m\in N_0$

and $\beta\in N$

$k=2,3,4$................

Then $f(z)\in\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$ iff it can be expressed in the form

$$f(z)=\sum_{k=2}^{\infty}\delta_{k+p-1}f_{k+p-1}(z)$$

Where $\delta_{k+p-1}\geq 0$ and $\sum_{k=1}^{\infty}\delta_{k+p-1}=1$

Proof Suppose that-

$$f(z)=\sum_{k=1}^{\infty}\delta_{k+p-1}f_{k+p-1}(z)$$

$$=f_p(z)-\sum_{k=1}^{\infty}\delta_{k+p-1}f_{k+p-1}(z)$$

$$=z^p-\frac{\delta_{k+p-1}(A-B)(1-\alpha)\,(p\beta)^n}{(1-B)\left[(p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right]+(A-B)(1-\alpha)\,(k+p+\beta-2)^n\mu_{k+p-1}(\beta)}z^{k+p+\beta-2}$$

Then from Theorem 2.2, we have-

$$\sum_{k=2}^{\infty}\Big\{(1-B)\left[\left((p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right)+(A-B)(1-\alpha)\,(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right.$$

$$\left.\times\frac{\delta_{k+p-1}(A-B)(1-\alpha)\,(p\beta)^n}{(1-B)\left[(p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right]+(A-B)(1-\alpha)\,(k+p+\beta-2)^n\mu_{k+p-1}(\beta)}\right\}$$

$$=(A-B)(1-\alpha)\,(p\beta)^n\sum_{k=2}^{\infty}\delta_{k+p-1}$$

$$=(A-B)(1-\alpha)\,(p\beta)^n\left(1-\delta_p\right)\leq(A-B)(1-\alpha)\,(p\beta)^n$$

Then $f(z)\in\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$ by Theorem 2.2

Conversely, suppose that $f(z)\in\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$

Since.

$$a_{k+p-1}(\beta) \le \frac{(A-B)(1-\alpha)\,(p\beta)^n}{(1-B)\left[(p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right]+(A-B)(1-\alpha)\,(k+p+\beta-2)^n\mu_{k+p-1}(\beta)}$$

$$\left|a_{k+p-1}(\beta)\right| \le \frac{(A-B)(1-\alpha)\,(p\beta)^n}{(1-B)\left[(p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right]+(A-B)(1-\alpha)\,(k+p+\beta-2)^n\mu_{k+p-1}(\beta)}$$

We say that -

$$\delta_{k+p-1} \le \frac{(1-B)\left[(p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right]+(A-B)(1-\alpha)\,(k+p+\beta-2)^n\mu_{k+p-1}(\beta)}{(A-B)(1-\alpha)\,(p\beta)^n}$$

$(\beta \in N, k = 2,3,4.......$

and $\delta_1 = 1-\sum_{k=2}^{\infty}\delta_{k+p-1}$

Then $f(z)= z^p-\sum_{k=2}^{\infty}\delta_{k+p-1}f_{k+p-1}(z)$

This complete the proof of the theorem

Corollary 3.1 Extreme points of the class

$f_p(z)\in z^p$

$$f_{k+p-1}(z)\in z^p-\frac{(A-B)(1-\alpha)\,(p\beta)^n}{(1-B)\left[(p\beta)^m-(p\beta)^n+(k+p+\beta-2)^m\lambda_{k+p-1}(\beta)-(k+p+\beta-2)^n\mu_{k+p-1}(\beta)\right]+(A-B)(1-\alpha)\,(k+p+\beta-2)^n\mu_{k+p-1}(\beta)}z^{k+p+\beta-2}$$

For some $\lambda_{k+p-1}(\beta)\ge 0,\ \mu_{k+p-1}(\beta)\ge 0,\ \lambda_{k+p-1}(\beta)\ge\mu_{k+p-1}(\beta),\ \alpha\,(0\le\alpha<1),\ m\in N_0$

and $\beta\in N$, $k=2,3,4................$

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